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Wei He^{*} Yeneng Sun[†]

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Abstract

The existence of stationary Markov perfect equilibria in stochastic games is shown in several contexts under a general condition called “coarser transition kernels”. These results include various earlier existence results on correlated equilibria, noisy stochastic games, stochastic games with mixtures of constant transition kernels as special cases. The minimality of the condition is illustrated. The results here also shed some new light on a recent example on the nonexistence of stationary equilibrium. The proofs are remarkably simple via establishing a new connection between stochastic games and conditional expectations of correspondences.

Keywords: Stochastic game, stationary Markov perfect equilibrium, equilibrium existence, coarser transition kernel.

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1 Introduction

Beginning with [Shapley \(1953\)](#), the existence of stationary Markov perfect equilibria in discounted stochastic games has remained an important problem. Given that stochastic games with general state spaces have found applications in various areas in economics, the issue on the existence of an equilibrium in stationary strategies for such games has received considerable attention in the last two decades. However, no general existence result, except for several special classes of stochastic games, has been obtained in the literature so far.¹

[Nowak and Raghavan \(1992\)](#) and [Duffie *et al.* \(1994\)](#) proved the existence of correlated stationary Markov perfect equilibria in stochastic games.² They essentially assumed that there is a randomization device publicly known to all players which is irrelevant to the fundamental parameters of the game. Stationary Markov perfect equilibria have been shown to exist by [Nowak \(2003\)](#) and [Duggan \(2012\)](#) for stochastic games with some special structures. [Nowak \(2003\)](#) studied a class of stochastic games in which the transition probability has a fixed countable set of atoms while its atomless part is a finite combination of atomless measures that do not depend on states and actions, i.e. a mixture of constant transition kernels. [Duggan \(2012\)](#) considered stochastic games with a specific product structure, namely stochastic games with noise – which is a history-irrelevant component of the state and could influence the payoff functions and transitions. Recently, [Levy \(2013\)](#) presented a counterexample showing that a stochastic game satisfying the usual conditions has no stationary Markov perfect equilibrium. This implies that a general existence result could only hold under some suitable conditions.

Our main purpose is to show the existence of stationary Markov perfect equilibria in stochastic games under a general condition called “(decomposable) coarser transition kernels” by establishing a new connection between the equilibrium payoff correspondences in stochastic games and a general result on the conditional expectations of correspondences. In a typical stochastic game with a general state space, there could be four sources of information that are generated respectively by the action correspondences, the stage payoffs, the transition probability itself and the transition kernel. As long as there is enough information in the first three sources that can not be covered by the information conveyed in the transition kernel, one would expect the total information that comes from the four possibly different sources to be essentially more than the information from the transition

¹We shall only discuss those papers which are the closest to our results here. For detailed discussions about the literature of stochastic games, see [Duffie *et al.* \(1994\)](#), [Duggan \(2012\)](#), [Levy \(2013\)](#), [Nowak and Raghavan \(1992\)](#), and their references.

²[Duffie *et al.* \(1994\)](#) obtained additional ergodic properties under stronger conditions.

kernel eventwise, which is exactly the condition of “coarser transition kernels”.³ When we do not have a coarser transition kernel, we can still work with the case of a “decomposable coarser transition kernel” in the sense that the transition kernel is decomposed as a sum of finitely many components with each component being the product of a “coarser” transition function and a density function.⁴

Theorem 1 below shows that under the condition of a coarser transition kernel, a stochastic game always has a stationary Markov perfect equilibrium.⁵ A very simple proof of that result is provided by introducing a convexity type result of [Dynkin and Evstigneev \(1976\)](#) on the conditional expectation of a correspondence to the existence problem. We point out that stochastic games with sunspot/noise have coarser transition kernels; and thus our result covers the existence results for such stochastic games while no product structure is imposed on the state space. We then consider the more general case with a decomposable coarser transition kernel and prove in Theorem 2 the existence of a stationary Markov perfect equilibrium. Proposition 1 extends to the case with atoms and presents an existence result that includes that of [Nowak \(2003\)](#) as a special case.⁶ We will also illustrate the minimality of our general condition from a technical point of view. Moreover, we analyze a recent nonexistence example in [Levy \(2013\)](#) and demonstrate how this specific game fails to satisfy our condition in Proposition 1.

The rest of the paper is organized as follows. Section 2 presents the model of discounted stochastic games. In Section 3, we propose the condition of a (decomposable) coarser transition kernel (on the atomless part) and prove the existence of stationary Markov perfect equilibria in several contexts. The minimality of this condition is also illustrated. In Section 4, we discuss the relationship between our results and several previous existence results. Section 5 concludes the paper.

2 Discounted Stochastic Game

Consider an m -person discounted stochastic game:

³It is worthwhile to point out that the consideration of such information gap arises naturally in economic models. For example, the geometric Brownian motion, which is widely used in asset pricing models, has strictly increasing information filtrations – the information at a previous time is always coarser than the information at a later time eventwise; see, for example, [Duffie \(2001, p.88\)](#). Note that the usual sample space of a geometric Brownian motion is the space of continuous functions endowed with the Wiener measure, which has no natural product structure.

⁴Such a density function is allowed to carry any information within the model. Thus, the transition kernel itself may have the possibility of carrying the full information in the model.

⁵We state our result for a completely general state space that does not require a complete separable metric structure as in some earlier work.

⁶[Levy \(2013, Example II\)](#) also includes an atom at 1 as an absorbing state.

- $I = \{1, \dots, m\}$ is the set of players.
- (S, \mathcal{S}) is a measurable space representing the states of nature.
- For each player $i \in I$, X_i is a nonempty compact metric space of actions with its Borel σ -algebra $\mathcal{B}(X_i)$. Let $X = \prod_{1 \leq i \leq m} X_i$, and $\mathcal{B}(X)$ the Borel σ -algebra on X .
- For each $i \in I$, A_i is a nonempty, \mathcal{S} -measurable, compact valued correspondence from S to X_i , $A_i(s)$ is the set of feasible actions for player i at state s . Let $A(s) = \prod_{i \in I} A_i(s)$ for each $s \in S$.
- For each $i \in I$, $u_i : S \times X \rightarrow \mathbb{R}$ is a stage-payoff with an absolute bound C (i.e., for all $i \in I$, $(s, x) \in S \times X$, $|u_i(s, x)| \leq C$ for some positive number C) such that $u_i(s, x)$ is \mathcal{S} -measurable in s for each $x \in X$ and continuous in x for each $s \in S$.
- $\beta_i \in [0, 1)$ is player i 's discount factor.
- $Q : S \times X \times \mathcal{S} \rightarrow [0, 1]$ is a transition probability representing the law of motion for the states.
 1. $Q(\cdot | s, x)$ (abbreviated as $Q_{(s, x)}$) is a probability measure on (S, \mathcal{S}) for all $s \in S$ and $x \in X$, and for all $E \in \mathcal{S}$, $Q(E | \cdot, \cdot)$ is $\mathcal{S} \otimes \mathcal{B}(X)$ -measurable.
 2. $Q(\cdot | s, x)$ is absolutely continuous with respect to λ for all s and x and $q(\cdot | s, x)$ (abbreviated as $q_{(s, x)}$) is the corresponding Radon-Nikodym derivative, where λ is a countably-additive probability measure on (S, \mathcal{S}) .
 3. For all $s \in S$, the mapping $q(\cdot | s, x)$ satisfies the following continuity condition in x : for any sequence $x^n \rightarrow x^0$,

$$\int_S |q(s_1 | s, x^n) - q(s_1 | s, x^0)| d\lambda(s_1) \rightarrow 0.$$

The game is played in discrete time and past history is observable by all the players. The game starts from some initial state. If s is the state at stage t and $x \in X$ is the action profile chosen simultaneously by the m players at this stage, then $Q(E | s, x)$ is the probability that the state at stage $t + 1$ belongs to the set E given s and x .

For a Borel set A in a complete separable metric space, $\mathcal{M}(A)$ is the set of all Borel probability measures on A . A strategy of player i is a measurable mapping f_i from the past history to $\mathcal{M}(X_i)$ which places probability 1 on the set of feasible actions. A stationary Markov strategy for player i is an \mathcal{S} -measurable mapping $f_i : S \rightarrow \mathcal{M}(X_i)$ such that $f_i(s)$ places probability 1 on the set $A_i(s)$ for each $s \in S$. Given a stationary Markov strategy f , the continuation values $v(\cdot, f)$ give an essentially bounded \mathcal{S} -measurable mapping from S to \mathbb{R}^m uniquely determined

by the following recursion

$$v_i(s, f) = \int_X \left[(1 - \beta_i) u_i(s, x) + \beta_i \int_S v_i(s_1, f) Q(ds_1 | s, x) \right] f(dx | s). \quad (1)$$

The strategy profile f is a **stationary Markov perfect equilibrium** if the discounted expected payoff of each player i is maximized by his strategy f_i in every state $s \in S$. By standard results in dynamic programming, it means that the continuation values v solve the following recursive maximization problem:

$$v_i(s, f) = \max_{x_i \in A_i(s)} \int_{X_{-i}} \left[(1 - \beta_i) u_i(s, x_i, x_{-i}) + \beta_i \int_S v_i(s_1, f) Q(ds_1 | s, x_i, x_{-i}) \right] f_{-i}(dx_{-i} | s), \quad (2)$$

where x_{-i} and X_{-i} have the usual meanings, and $f_{-i}(s)$ is the product probability $\otimes_{j \neq i} f_j(s)$ on the set of actions of all players other than i at the state s .

3 Main Results

In this section, we will show the existence of stationary Markov perfect equilibria in stochastic games with a general condition called “coarser transition kernels”. We use this condition in three different contexts. The minimality of the condition is also demonstrated in the last subsection.

3.1 Stochastic games with coarser transition kernels

We follow the notation in Section 2; and assume that the probability measure λ on the measurable state space (S, \mathcal{S}) is atomless. Let \mathcal{G} be a sub- σ -algebra of \mathcal{S} . For any nonnegligible set $D \in \mathcal{S}$, let \mathcal{G}^D and \mathcal{S}^D be the respective σ -algebras $\{D \cap D' : D' \in \mathcal{G}\}$ and $\{D \cap D' : D' \in \mathcal{S}\}$ on D . A set $D \in \mathcal{S}$ is said to be a \mathcal{G} -atom if $\lambda(D) > 0$ and given any $D_0 \in \mathcal{S}^D$, there exists a set $D_1 \in \mathcal{G}^D$ such that $\lambda(D_0 \Delta D_1) = 0$. For convenience, one often considers the strong completion of \mathcal{G} in \mathcal{S} , whose sets have the form $E \Delta E_0$ with $E \in \mathcal{G}$ and E_0 a null set in \mathcal{S} . When \mathcal{G} is strongly completed, D is a \mathcal{G} -atom if and only if \mathcal{G}^D and \mathcal{S}^D are identical.

Definition 1. A discounted stochastic game is said to have a **coarser transition kernel** if for some sub- σ -algebra \mathcal{G} of \mathcal{S} , $q(\cdot | s, x)$ is \mathcal{G} -measurable for all $s \in S$ and $x \in X$, and \mathcal{S} has no \mathcal{G} -atom.

The sub- σ -algebra \mathcal{G} of \mathcal{S} can be regarded as the σ -algebra generated by the transition kernel $q(\cdot | s, x)$ for all $s \in S$ and $x \in X$. Let \mathcal{S}_A , \mathcal{S}_u and \mathcal{S}_Q be the

sub- σ -algebras of \mathcal{S} that are generated respectively by the action correspondences $A_i(\cdot)$ for all $i \in I$, the stage payoffs $u_i(\cdot, x)$ for all $i \in I$ and $x \in X$, and the transition probability $Q(E|\cdot, x)$ for all $E \in \mathcal{S}$ and $x \in X$.⁷ The sub- σ -algebras \mathcal{S}_A , \mathcal{S}_u and \mathcal{S}_Q can be viewed as the information carried respectively by the action correspondences, the stage payoffs and the transition probability. The σ -algebra \mathcal{S} , which contains \mathcal{S}_A , \mathcal{S}_u , \mathcal{S}_Q and \mathcal{G} , represents the total information available for strategies.⁸ If \mathcal{S} has a \mathcal{G} -atom D , then \mathcal{S} will coincide with \mathcal{G} (modulo null sets if necessary) when restricted to D . Thus, the condition of coarser transition kernel simply means that the total information available for strategies is more than the information conveyed in the transition kernel on any non-trivial event. As long as there is enough different information in the action correspondences, stage payoffs and transition probability which can not be covered by the information from the transition kernel, one would expect the total information in \mathcal{S} that comes from four possibly different sources to be strictly more than that in \mathcal{G} eventwise. This is exactly our condition.

Theorem 1. *Every discounted stochastic game with a coarser transition kernel has a stationary Markov perfect equilibrium.*

Let $L_1^{\mathcal{S}}(S, \mathbb{R}^m)$ and $L_{\infty}^{\mathcal{S}}(S, \mathbb{R}^m)$ be the L_1 and L_{∞} spaces of all \mathcal{S} -measurable mappings from S to \mathbb{R}^m with the usual norm; that is,

$$L_1^{\mathcal{S}}(S, \mathbb{R}^m) = \{f: f \text{ is } \mathcal{S}\text{-measurable and } \int_S \|f\| d\lambda < \infty\},$$

$$L_{\infty}^{\mathcal{S}}(S, \mathbb{R}^m) = \{f: f \text{ is } \mathcal{S}\text{-measurable and essentially bounded under } \lambda\},$$

where $\|\cdot\|$ is the usual norm in \mathbb{R}^m . By the Riesz representation theorem (see Theorem 13.28 of [Aliprantis and Border \(2006\)](#)), $L_{\infty}^{\mathcal{S}}(S, \mathbb{R}^m)$ can be viewed as the dual space of $L_1^{\mathcal{S}}(S, \mathbb{R}^m)$. Then $L_{\infty}^{\mathcal{S}}(S, \mathbb{R}^m)$ is a locally convex, Hausdorff topological vector space under the weak* topology. Suppose that V is a subset of $L_{\infty}^{\mathcal{S}}(S, \mathbb{R}^m)$ such that for any $v \in V$, $\|v\|_{\infty} \leq C$, where C is an upper bound of the stage payoff function u . Then V is nonempty and convex. Moreover, V is compact under the weak* topology by Alaoglu's Theorem (see Theorem 6.21 of [Aliprantis and Border \(2006\)](#)).

Given any $v = (v_1, \dots, v_m) \in V$ and $s \in S$, we consider the game $\Gamma(v, s)$. The action space for player i is $A_i(s)$. The payoff of player i with the action profile

⁷Note that the information generated by the transition probability and the information generated by the transition kernel could be different.

⁸That is, the strategies must be \mathcal{S} -measurable.

$x \in A(s)$ is given by

$$U_i(s, x)(v) = (1 - \beta_i)u_i(s, x) + \beta_i \int_S v_i(s_1)Q(ds_1|s, x). \quad (3)$$

A mixed strategy for player i is an element in $\mathcal{M}(A_i(s))$, and a mixed strategy profile is an element in $\bigotimes_{i \in I} \mathcal{M}(A_i(s))$. The set of mix strategy Nash equilibria of the static game $\Gamma(v, s)$, denoted by $N(v, s)$, is a nonempty compact subset of $\bigotimes_{i \in I} \mathcal{M}(X_i)$ under the weak* topology. Let $P(v, s)$ be the set of payoff vectors induced by the Nash equilibria in $N(v, s)$, and $\text{co}(P)$ the convex hull of P . Then $\text{co}(P)$ is a correspondence from $V \times S$ to \mathbb{R}^m . Let $R(v)$ (resp. $\text{co}(R(v))$) be the set of λ -equivalence classes of \mathcal{S} -measurable selections of $P(v, \cdot)$ (resp. $\text{co}(P(v, \cdot))$) for each $v \in V$.

By the standard argument, one can show that for each $v \in V$, $P(v, \cdot)$ is \mathcal{S} -measurable and compact valued, and $\text{co}(R(v))$ is nonempty, convex, weak* compact valued and upper hemicontinuous (see, for example, Lemmas 6 and 7 in [Nowak and Raghavan \(1992\)](#)). Then the correspondence $\text{co}(R) : V \rightarrow V$ maps the nonempty, convex, weak* compact set V (a subset of a locally convex Hausdorff topological vector space) to nonempty, convex subsets of V , and it has a closed graph in the weak* topology. By the classical Fan-Glicksberg Fixed Point Theorem, there is a fixed point $v' \in V$ such that $v' \in \text{co}(R)(v')$. That is, v' is an \mathcal{S} -measurable selection of $\text{co}(P(v', \cdot))$.

For any integrably bounded correspondence G from S to \mathbb{R}^m , define

$$\mathcal{I}_G^{(\mathcal{S}, \mathcal{G})} = \{E(g|\mathcal{G}) : g \text{ is an } \mathcal{S}\text{-measurable selection of } G\}.$$

The conditional expectation is taken with respect to the probability measure λ .

The following lemma is due to [Dynkin and Evstigneev \(1976, Theorem 1.2\)](#).

Lemma 1. *If \mathcal{S} has no \mathcal{G} -atom,⁹ then for any \mathcal{S} -measurable, integrably bounded, closed valued correspondence G , $\mathcal{I}_G^{(\mathcal{S}, \mathcal{G})} = \mathcal{I}_{\text{co}(G)}^{(\mathcal{S}, \mathcal{G})}$.*

Proof of Theorem 1. Given v' , $\mathcal{I}_{P_{v'}}^{(\mathcal{S}, \mathcal{G})} = \mathcal{I}_{\text{co}(P_{v'})}^{(\mathcal{S}, \mathcal{G})}$ by Lemma 1. There exists an \mathcal{S} -measurable selection v^* of $P_{v'}$ such that $E(v^*|\mathcal{G}) = E(v'|\mathcal{G})$. For each $i \in I$, $s \in S$ and $x \in X$, we have

$$\int_S v_i^*(s_1)Q(ds_1|s, x) = \int_S v_i^*(s_1)q(s_1|s, x) d\lambda(s_1) = \int_S E(v_i^*q_{(s, x)}|\mathcal{G}) d\lambda$$

⁹In [Dynkin and Evstigneev \(1976\)](#) (DE), a set $D \in \mathcal{S}$ is said to be a \mathcal{G} -atom if $\lambda(D) > 0$ and given any $D_0 \in \mathcal{S}^D$, $\lambda(s \in S : 0 < \lambda(D_0 | \mathcal{G})(s) < \lambda(D | \mathcal{G})(s)) = 0$. It is clear that if D is a \mathcal{G} -atom in the sense of (DE), then D is a \mathcal{G} -atom in our sense. If D is a \mathcal{G} -atom in the sense of (DE), fix an arbitrary set $D_0 \in \mathcal{S}^D$, let $E = \{s : \lambda(D_0 | \mathcal{G})(s) = \lambda(D | \mathcal{G})(s)\}$. Then $E \in \mathcal{G}$ and $\lambda(D_0 | \mathcal{G}) = \lambda(D | \mathcal{G})\mathbf{1}_E = \lambda(D \cap E | \mathcal{G})$ for λ -almost all $s \in S$, where $\mathbf{1}_E$ is the indicator function of E . It is easy to see $\lambda(D_0 \Delta (D \cap E)) = 0$, which implies that D is a \mathcal{G} -atom in our sense.

$$\begin{aligned}
&= \int_S E(v_i^* | \mathcal{G}) q_{(s,x)} d\lambda = \int_S E(v_i' | \mathcal{G}) q_{(s,x)} d\lambda = \int_S E(v_i' q_{(s,x)} | \mathcal{G}) d\lambda \\
&= \int_S v_i'(s_1) q(s_1 | s, x) d\lambda = \int_S v_i'(s_1) Q(ds_1 | s, x).
\end{aligned}$$

By Equation (3), $\Gamma(v^*, s) = \Gamma(v', s)$ for any $s \in S$, and hence $P(v^*, s) = P(v', s)$. Thus, v^* is an \mathcal{S} -measurable selection of P_{v^*} .

By the definition of P_{v^*} , there exists an \mathcal{S} -measurable mapping f^* from S to $\bigotimes_{i \in I} \mathcal{M}(X_i)$ such that $f^*(s)$ is a mixed strategy Nash equilibrium of the game $\Gamma(v^*, s)$ and $v^*(s)$ is the corresponding equilibrium payoff for each $s \in S$. It is clear that Equations (1) and (2) hold for v^* and f^* , which implies that f^* is a stationary Markov perfect equilibrium. \square

3.2 Stochastic games with decomposable coarser transition kernels

As in Subsection 3.1, we follow the notation in Section 2 and assume the probability measure λ on the measurable state space (S, \mathcal{S}) to be atomless. In this subsection, we will relax the assumption in Subsection 3.1 that the transition kernel q is measurable with respect to the sub- σ -algebra \mathcal{G} . We will allow the transition kernel q itself to be \mathcal{S} -measurable, but require q to be decomposed as a sum of J components with each component being the product of a \mathcal{G} -measurable transition function and an \mathcal{S} -measurable function. A stationary Markov perfect equilibrium still exists in such a case.

Definition 2. A discounted stochastic game is said to have a **decomposable coarser transition kernel** if \mathcal{S} has no \mathcal{G} -atom and for some positive integer J , $q(s_1 | s, x) = \sum_{1 \leq j \leq J} q_j(s_1, s, x) \rho_j(s_1)$, where q_j is product measurable and $q_j(\cdot, s, x)$ is \mathcal{G} -measurable for each $s \in S$ and $x \in X$, q_j and ρ_j are all nonnegative, and ρ_j is integrable on the atomless probability space $(S, \mathcal{S}, \lambda)$, $j = 1, \dots, J$.

Note that when a discounted stochastic game has a decomposable coarser transition kernel, the collection of mappings $\{q(\cdot | s, x)\}_{s \in S, x \in X}$ themselves may not be \mathcal{G} -measurable since the ρ_j for $1 \leq j \leq J$ are required to be \mathcal{S} -measurable.

Theorem 2. Every discounted stochastic game with a decomposable coarser transition kernel has a stationary Markov perfect equilibrium.

Proof. Following the same argument and notation as in Subsection 3.1, there is a mapping $v' \in V$ such that $v' \in \text{co}(R(v'))$. Let $H(s) = \{(a, a \cdot \rho_1(s), \dots, a \cdot \rho_J(s)) : a \in P_{v'}(s)\}$, and $\text{co}(H(s))$ the convex hull of $H(s)$ for each $s \in S$. It is clear that H is \mathcal{S} -measurable, integrably bounded and closed valued. Then

$\mathcal{I}_H^{(\mathcal{S}, \mathcal{G})} = \mathcal{I}_{\text{co}(H)}^{(\mathcal{S}, \mathcal{G})}$ by Lemma 1, which implies that there exists an \mathcal{S} -measurable selection v^* of $P_{v'}$ such that $E(v^* \rho_j | \mathcal{G}) = E(v' \rho_j | \mathcal{G})$ for each $1 \leq j \leq J$. For each $i \in I$, $s \in S$ and $x \in X$, we have

$$\begin{aligned}
\int_S v_i^*(s_1) Q(ds_1 | s, x) &= \sum_{1 \leq j \leq J} \int_S v_i^*(s_1) \cdot q_j(s_1, s, x) \cdot \rho_j(s_1) d\lambda(s_1) \\
&= \sum_{1 \leq j \leq J} \int_S E(v_i^* \rho_j | \mathcal{G})(s_1) \cdot q_j(s_1, s, x) d\lambda(s_1) \\
&= \sum_{1 \leq j \leq J} \int_S E(v'_j \rho_j | \mathcal{G})(s_1) \cdot q_j(s_1, s, x) d\lambda(s_1) \\
&= \sum_{1 \leq j \leq J} \int_S v'_j(s_1) \cdot q_j(s_1, s, x) \cdot \rho_j(s_1) d\lambda(s_1) \\
&= \int_S v'_i(s_1) Q(ds_1 | s, x).
\end{aligned}$$

By repeating the argument in the last paragraph of the proof for Theorem 1, we can obtain the existence of a stationary Markov perfect equilibrium. \square

3.3 Decomposable coarser transition kernels on the atomless part

In Theorems 1 and 2, we assume that the probability measure λ is atomless on (S, \mathcal{S}) . Below we shall consider the more general case that λ may have atoms. To guarantee the existence of stationary Markov perfect equilibria, we still assume the condition of decomposable coarser transition kernel, but now on the atomless part.

1. There exist disjoint \mathcal{S} -measurable subsets S_1 and S_2 such that $S_1 \cup S_2 = S$, $\lambda|_{S_1}$ is the atomless part of λ while $\lambda|_{S_2}$ is the purely atomic part of λ . The subset S_2 is countable and each point in S_2 is \mathcal{S} -measurable.¹⁰
2. For $s_1 \in S_1$, the transition kernel $q(s_1 | s, x) = \sum_{1 \leq j \leq J} q_j(s_1, s, x) \rho_j(s_1)$ for some positive integer J , and for $s \in S$ and $x \in X$, where q_j is nonnegative and product measurable, and ρ_j is nonnegative and integrable on the atomless measure space $(S_1, \mathcal{S}^{S_1}, \lambda^{S_1})$, $j = 1, \dots, J$.¹¹

Remark 1. *By the continuity condition on the transition kernel, one can easily*

¹⁰This assumption is only for simplicity. One can easily consider the case that S_2 is a collection of at most countably many atoms.

¹¹It is clear that for any $E \in \mathcal{S}$, the transition probability $Q(E | s, x) = \int_{E \cap S_1} q(s_1 | s, x) d\lambda(s_1) + \sum_{s_2 \in S_2} \mathbf{1}_E(s_2) q(s_2 | s, x) \lambda(s_2)$ for any $s \in S$ and $x \in X$.

deduce that for all $s \in S$ and any sequence $x^n \rightarrow x^0$,

$$\int_{S_1} |q(s_1|s, x^n) - q(s_1|s, x^0)| d\lambda(s_1) \rightarrow 0.$$

$$|q(s_2|s, x^n) - q(s_2|s, x^0)| \rightarrow 0$$

for any $s_2 \in S_2$ such that $\lambda(s_2) > 0$.

Definition 3. Let \mathcal{G} be a sub- σ -algebra of \mathcal{S}^{S_1} . A discounted stochastic game is said to have a **decomposable coarser transition kernel on the atomless part** if \mathcal{S}^{S_1} has no \mathcal{G} -atom under λ and $q_j(\cdot, s, x)$ is \mathcal{G} -measurable on S_1 for each $s \in S$ and $x \in X$, $j = 1, \dots, J$.

Proposition 1. Every discounted stochastic game with a decomposable coarser transition kernel on the atomless part has a stationary Markov perfect equilibrium.

Proof. Let V_1 be the set of λ -equivalence classes of \mathcal{S} -measurable mappings from S_1 to \mathbb{R}^m bounded by C . For each $i \in I$, let F_i be the set of all $f_i: S_2 \rightarrow \mathcal{M}(X_i)$ such that $f_i(s)(A_i(s)) = 1$ for all $s \in S_2$, $F = \prod_{i \in I} F_i$. Let V_2 be the set of mappings from S_2 to \mathbb{R}^m bounded by C ; V_2 is endowed with the supremum metric and hence a complete metric space.

Given $s \in S$, $v^1 \in V_1$ and $v^2 \in V_2$, consider the game $\Gamma(v^1, v^2, s)$. The action space for player i is $A_i(s)$. The payoff of player i with the action profile $x \in A(s)$ is given by

$$\begin{aligned} \Phi_i(s, x, v^1, v^2) &= (1 - \beta_i)u_i(s, x) + \beta_i \sum_{1 \leq j \leq J} \int_{S_1} v_i^1(s_1) q_j(s_1|s, x) \rho_j(s_1) d\lambda(s_1) \\ &\quad + \beta_i \sum_{s_2 \in S_2} v_i^2(s_2) q(s_2|s, x) \lambda(s_2). \end{aligned} \quad (4)$$

The set of mixed strategy Nash equilibria in the game $\Gamma(v^1, v^2, s)$ is denoted as $N(v^1, v^2, s)$. Let $P(v^1, v^2, s)$ be the set of payoff vectors induced by the Nash equilibria in $N(v^1, v^2, s)$, and $\text{co}(P)$ the convex hull of P .

Given $v^1 \in V_1$, $f \in F$, define a mapping Π from V_2 to V_2 such that for each $i \in I$, $v^2 \in V_2$ and $s_2 \in S_2$,

$$\Pi_i(f_{-i}, v^1)(v^2)(s_2) = \max_{\phi_i \in F_i} \int_{X_{-i}} \int_{X_i} \Phi_i(s_2, x_i, x_{-i}, v^1, v^2) \phi_i(dx_i|s_2) f_{-i}(dx_{-i}|s_2). \quad (5)$$

Let $\beta = \max\{\beta_i: i \in I\}$. Then for any $v^1 \in V_1$, $v^2, \bar{v}^2 \in V_2$, $x \in X$ and $s \in S_2$,

$$\begin{aligned} |\Phi_i(s, x, v^1, v^2) - \Phi_i(s, x, v^1, \bar{v}^2)| &\leq \beta_i \sum_{s_2 \in S_2} |v_i^2(s_2) - \bar{v}_i^2(s_2)| q(s_2|s, x) \lambda(s_2) \\ &\leq \beta_i \sup_{s_2 \in S_2} |v_i^2(s_2) - \bar{v}_i^2(s_2)| \leq \beta \sup_{s_2 \in S_2} |v_i^2(s_2) - \bar{v}_i^2(s_2)|. \end{aligned}$$

Thus, Π is a β -contraction mapping. There is a unique $\bar{v}^2 \in V_2$ such that $\Pi_i(f_{-i}, v^1)(\bar{v}^2)(s_2) = \bar{v}_i^2(s_2)$ for each $i \in I$ and $s_2 \in S_2$. Let $W(v^1, f)$ be the set of all $\phi \in F$ such that for each $i \in I$ and $s_2 \in S_2$,

$$\bar{v}_i^2(s_2) = \int_{X_{-i}} \int_{X_i} \Phi_i(s_2, x_i, x_{-i}, v^1, \bar{v}^2) \phi_i(dx_i|s_2) f_{-i}(dx_{-i}|s_2). \quad (6)$$

Let $\text{co}(R(v^1, f))$ be the set of λ -equivalence classes of \mathcal{S} -measurable selections of $\text{co}(P(v^1, \bar{v}^2, \cdot))$ restricted to S_1 , where \bar{v}^2 is generated by v^1 and f as above. Denote $\Psi(v^1, f) = \text{co}(R(v^1, f)) \times W(v^1, f)$ for each $v^1 \in V_1$ and $f \in F$.

By the standard argument, one can show that Ψ is convex, compact valued and upper hemicontinuous (see, for example, [Nowak \(2003\)](#)). By Fan-Glicksberg's Fixed Point Theorem, Ψ has a fixed point $(v^{1'}, f^{2'}) \in V_1 \times F$. Let $v^{2'}$ be the mapping from S_2 to \mathbb{R}^m that is generated by $v^{1'}$ and $f^{2'}$ through the β -contraction mapping Π as above. Then, $v^{1'}$ is an \mathcal{S} -measurable selection of $\text{co}(P(v^{1'}, v^{2'}, \cdot))$ restricted to S_1 ; and furthermore we have for each $i \in I$ and $s_2 \in S_2$,

$$v_i^{2'}(s_2) = \int_{X_{-i}} \int_{X_i} \Phi_i(s_2, x_i, x_{-i}, v^{1'}, v^{2'}) f_i^{2'}(dx_i|s_2) f_{-i}^{2'}(dx_{-i}|s_2), \quad (7)$$

$$\Pi_i(f_{-i}^{2'}, v^{1'})(v^{2'})(s_2) = v_i^{2'}(s_2). \quad (8)$$

Following the same argument as in the proof of Theorem 2, one can show that there exists an \mathcal{S} -measurable selection v^{1*} of $P_{(v^{1'}, v^{2'})}$ such that $E(v^{1*} \rho_j | \mathcal{G}) = E(v^{1'} \rho_j | \mathcal{G})$ for each $1 \leq j \leq J$, where the conditional expectation is taken on $(S_1, \mathcal{S}^{S_1}, \lambda^{S_1})$ with λ^{S_1} the normalized probability measure on (S_1, \mathcal{S}^{S_1}) . Hence, for any $s \in S$ and $x \in A(s)$, $\Phi_i(s, x, v^{1'}, v^{2'}) = \Phi_i(s, x, v^{1*}, v^{2'})$, $\Gamma(v^{1'}, v^{2'}, s) = \Gamma(v^{1*}, v^{2'}, s)$, and therefore $P(v^{1'}, v^{2'}, s) = P(v^{1*}, v^{2'}, s)$. Thus, v^{1*} is an \mathcal{S} -measurable selection of $P_{(v^{1*}, v^{2'})}$, and there exists an \mathcal{S} -measurable mapping $f^{1*}: S_1 \rightarrow \bigotimes_{i \in I} \mathcal{M}(X_i)$ such that $f^{1*}(s)$ is a mixed strategy equilibrium of the game $\Gamma(v^{1*}, v^{2'}, s)$ and $v^{1*}(s)$ the corresponding equilibrium payoff for each $s \in S_1$.

Let $v^*(s)$ be $v^{1*}(s)$ for $s \in S_1$ and $v^{2'}(s)$ for $s \in S_2$. Similarly, let $f^*(s)$ be $f^{1*}(s)$ for $s \in S_1$ and $f^{2'}(s)$ for $s \in S_2$. For $s_1 \in S_1$, since v^{1*} is a measurable selection of $P_{(v^{1*}, v^{2'})}$ on S_1 , the equilibrium property of $f^{1*}(s_1)$ then implies that Equations (1) and (2) hold for v^* and f^* . Next, for $s_2 \in S_2$, the identity $\Phi_i(s_2, x, v^{1'}, v^{2'}) = \Phi_i(s_2, x, v^{1*}, v^{2'})$ implies that Equations (7) and (8) still hold when $v^{1'}$ is replaced by v^{1*} , which means that Equations (1) and (2) hold for v^* and f^* . Therefore, f^* is a stationary Markov perfect equilibrium. \square

3.4 Minimality of the condition

In the previous three subsections, we show the existence of stationary Markov perfect equilibria in discounted stochastic games by assuming the condition of a (decomposable) coarser transition kernel (on the atomless part). This raises the question of whether our condition is minimal and, if so, then in what sense.

The central difficulty in the existence argument for stochastic games is typically due to the failure of the fixed-point method. As shown in Subsection 3.1, the correspondence R , which is the collection of selections from the equilibrium payoff correspondence P , will live in an infinite-dimensional space if there is a continuum of states. Thus, the desirable closedness and upper hemicontinuity properties would fail even though P has these properties. To handle such issues, the main approach in the literature is to work with the convex hull $\text{co}(R)$. We bypass this imposed convexity restriction by using the result that $\mathcal{I}_G^{(\mathcal{S}, \mathcal{G})} = \mathcal{I}_{\text{co}(G)}^{(\mathcal{S}, \mathcal{G})}$ for any \mathcal{S} -measurable, integrably bounded, closed valued correspondence G provided that \mathcal{S} has no \mathcal{G} -atom. Moreover, for the condition of a decomposable coarser transition kernel (on the atomless part), we assume that the transition kernel can be divided into finitely many parts. The following propositions demonstrate the minimality of our condition.

Proposition 2. *Suppose that $(S, \mathcal{S}, \lambda)$ has a \mathcal{G} -atom D with $\lambda(D) > 0$. Then there exists a measurable correspondence G from $(S, \mathcal{S}, \lambda)$ to $\{0, 1\}$ such that $\mathcal{I}_G^{(\mathcal{S}, \mathcal{G})} \neq \mathcal{I}_{\text{co}(G)}^{(\mathcal{S}, \mathcal{G})}$.*

Proof. Define a correspondence $G(s) = \begin{cases} \{0, 1\} & s \in D; \\ \{0\} & s \notin D. \end{cases}$ We claim that $\mathcal{I}_G^{(\mathcal{S}, \mathcal{G})} \neq$

$\mathcal{I}_{\text{co}(G)}^{(\mathcal{S}, \mathcal{G})}$. Let $g_1(s) = \frac{1}{2}\mathbf{1}_D$, where $\mathbf{1}_D$ is the indicator function of the set D . Then g_1 is an \mathcal{S} -measurable selection of $\text{co}(G)$. If there is an \mathcal{S} -measurable selection g_2 of G such that $E(g_1|\mathcal{G}) = E(g_2|\mathcal{G})$, then there is a subset $D_2 \subseteq D$ such that $g_2(s) = \mathbf{1}_{D_2}$. Since D is a \mathcal{G} -atom, for any \mathcal{S} -measurable subset $E \subseteq D$, there is a subset $E_1 \in \mathcal{G}$ such that $\lambda(E \triangle (E_1 \cap D)) = 0$. Then

$$\begin{aligned} \lambda(E \cap D_2) &= \int_S \mathbf{1}_E(s) g_2(s) d\lambda(s) = \int_S E(\mathbf{1}_{E_1} \mathbf{1}_D g_2 | \mathcal{G}) d\lambda = \int_S \mathbf{1}_{E_1} E(g_2 | \mathcal{G}) d\lambda \\ &= \int_S \mathbf{1}_{E_1} E(g_1 | \mathcal{G}) d\lambda = \frac{1}{2} \int_S \mathbf{1}_{E_1} \mathbf{1}_D d\lambda = \frac{1}{2} \lambda(E_1 \cap D) = \frac{1}{2} \lambda(E). \end{aligned}$$

Thus, $\lambda(D_2) = \frac{1}{2} \lambda(D) > 0$ by choosing $E = D$. However, $\lambda(D_2) = \frac{1}{2} \lambda(D_2)$ by choosing $E = D_2$, which implies that $\lambda(D_2) = 0$, a contradiction. \square

The key result that we need in the proof of Theorem 2 is $\mathcal{I}_H^{(\mathcal{S}, \mathcal{G})} = \mathcal{I}_{\text{co}(H)}^{(\mathcal{S}, \mathcal{G})}$. The question is whether a similar result holds if we generalize the condition of

a decomposable coarser transition kernel from a finite sum to a countable sum. We will show that this is not possible. Let $(S, \mathcal{S}, \lambda)$ be the Lebesgue unit interval (L, \mathcal{B}, η) . Suppose that $\{\varrho_n\}_{n \geq 0}$ is a complete orthogonal system in $L_2(S, \mathcal{S}, \lambda)$ such that ϱ_n takes value in $\{-1, 1\}$ and $\int_S \varrho_n d\lambda = 0$ for each $n \geq 1$ and $\varrho_0 \equiv 1$. Let $\rho_n = \varrho_n + 1$ for each $n \geq 1$ and $\rho_0 = \varrho_0$. Let $\{E_n\}_{n \geq 0}$ be a countable measurable partition of S and $q_n(s) = \mathbf{1}_{E_n}$ for each $n \geq 0$. Suppose that a transition kernel q is decomposed into a countable sum $q(s_1|s, x) = \sum_{n \geq 0} q_n(s) \rho_n(s_1)$. The following proposition shows that the argument for the case that J is finite is not valid for such an extension.¹²

Proposition 3. *There exists a correspondence G and a selection f of $\text{co}(G)$ such that for any σ -algebra $\mathcal{G} \subseteq \mathcal{S}$, there is no selection g of G with $E(g\rho_n|\mathcal{G}) = E(f\rho_n|\mathcal{G})$ for any $n \geq 0$.*

Proof. Let $G(s) = \{-1, 1\}$ and $f(s) = 0$ for all $s \in S$. Then f is a selection of $\text{co}(G)$. We claim that there does not exist an \mathcal{S} -measurable selection g of G such that $E(g\rho_n|\mathcal{G}) = E(f\rho_n|\mathcal{G})$ for any $n \geq 0$.

We show this by way of contradiction. Suppose that there exists an \mathcal{S} -measurable selection g of G such that $E(g\rho_n|\mathcal{G}) = 0$ for any $n \geq 0$. Then there exists a set $E \in \mathcal{S}$ such that $g(s) = \begin{cases} 1 & s \in E; \\ -1 & s \notin E. \end{cases}$ Thus, $\lambda(E) - \lambda(E^c) = \int_S g\rho_0 d\lambda = \int_S E(g\rho_0|\mathcal{G}) d\lambda = 0$, which implies $\lambda(E) = \frac{1}{2}$. Moreover, $\int_S g\varrho_n d\lambda = \int_S g\rho_n d\lambda - \int_S g d\lambda = \int_S E(g\rho_n|\mathcal{G}) d\lambda - 0 = 0$ for each $n \geq 1$, which contradicts the condition that $\{\varrho_n\}_{n \geq 0}$ is a complete orthogonal system. \square

Thus, our condition is minimal in the sense that, if one would like to adopt the measure-theoretical approach as used here to obtain a stationary Markov perfect equilibrium, then it is the most general condition.

4 Discussion

In this section, we shall discuss the relationship between our results and several related results.¹³

Correlated equilibria

It is proved in [Nowak and Raghavan \(1992\)](#) that a correlated stationary Markov perfect equilibrium exists in discounted stochastic games in the setup described in

¹²It is a variant of a well known example of Lyapunov.

¹³As mentioned in Footnote 1 in the introduction, we only consider those results that are most relevant to ours.

our Section 2. Duffie *et al.* (1994) obtained ergodic properties of such correlated equilibria under stronger conditions. They essentially assumed that players can observe the outcome of a public randomization device before making decisions at each stage.¹⁴ Thus, the new state space can be regarded as $S' = S \times L$ endowed with the product σ -algebra $\mathcal{S}' = \mathcal{S} \otimes \mathcal{B}$ and product measure $\lambda' = \lambda \otimes \eta$, where L is the unit interval endowed with the Borel σ -algebra \mathcal{B} and Lebesgue measure η . Denote $\mathcal{G}' = \mathcal{S} \otimes \{\emptyset, L\}$. Given $s', s'_1 \in S'$ and $x \in X$, the new transition kernel $q'(s'_1|s', x) = q(s_1|s, x)$, where s (resp. s_1) is the projection of s' (resp. s'_1) on S and q is the original transition kernel with the state space S . Thus, $q'(\cdot|s', x)$ is measurable with respect to \mathcal{G}' for any $s' \in S'$ and $x \in X$. It is obvious that S' has no \mathcal{G}' -atom. Then the condition of coarser transition kernel is satisfied for the extended state space $(S', \mathcal{S}', \lambda')$, and the existence of a stationary Markov perfect equilibrium follows from Theorem 1. The drawback of this approach is that the “sunspot” is irrelevant irrelevant to the fundamental parameters of the game. Our result shows that it can indeed enter the stage payoff u , the correspondence of feasible actions A and the transition probability Q .

Decomposable constant transition kernels on the atomless part

Nowak (2003) considered stochastic games with transition probabilities as combinations of finitely many measures on the atomless part. In particular, the structure of the transition probability in Nowak (2003) is as follows.

1. S_2 is a countable subset of S and $S_1 = S \setminus S_2$, each point in S_2 is \mathcal{S} -measurable.
2. There are atomless nonnegative measures μ_j concentrated on S_1 , nonnegative measures δ_k concentrated on S_2 , and measurable functions $q_j, b_k : S \times X \rightarrow [0, 1]$, $1 \leq j \leq J$ and $1 \leq k \leq K$, where J and K are positive integers. The transition probability $Q(\cdot|s, x) = \delta(\cdot|s, x) + Q'(\cdot|s, x)$ for each $s \in S$ and $x \in X$, where $\delta(\cdot|s, x) = \sum_{1 \leq k \leq K} b_k(s, x)\delta_k(\cdot)$ and $Q'(\cdot|s, x) = \sum_{1 \leq j \leq J} q_j(s, x)\mu_j(\cdot)$.
3. For any j and k , $q_j(s, \cdot)$ and $b_k(s, \cdot)$ are continuous on X for any $s \in S$.

We shall show that any stochastic game with the above structure satisfies the condition of decomposable coarser transition kernel on the atomless part.

Without loss of generality, assume that μ_j and δ_k are all probability measures. Let $\lambda(E) = \frac{1}{J+K}(\sum_{1 \leq j \leq J} \mu_j(E) + \sum_{1 \leq k \leq K} \delta_k(E))$ for any $E \in \mathcal{S}$. Then μ_j is absolutely continuous with respect to λ and assume that ρ_j is the Radon-Nikodym derivative for $1 \leq j \leq J$.

¹⁴For detailed discussions on such a public randomization device, or “sunspot”, see Duffie *et al.* (1994) and their references.

Given any $s \in S$ and $x \in X$, let

$$q(s'|s, x) = \begin{cases} \sum_{1 \leq j \leq J} q_j(s, x) \rho_j(s'), & \text{if } s' \in S_1; \\ \frac{\delta(s'|s, x)}{\lambda(s')}, & \text{if } s' \in S_2 \text{ and } \lambda(s') > 0; \\ 0, & \text{if } s' \in S_2 \text{ and } \lambda(s') = 0. \end{cases}$$

Then $Q(\cdot|s, x)$ is absolutely continuous with respect to λ and $q(\cdot|s, x)$ is the transition kernel. It is obvious that the condition of a decomposable coarser transition kernel on the atomless part is satisfied with $\mathcal{G} = \{\emptyset, S_1\}$. Then a stationary Markov perfect equilibrium exists by Proposition 1.

Noisy stochastic games

Duggan (2012) proved the existence of stationary Markov perfect equilibria in stochastic games with noise – a component of the state that is nonatomically distributed and not directly affected by the previous period's state and actions. The exogenously given product structure of the state space as considered by Duggan (2012) is defined as follows:

1. The set of states can be decomposed as $S = H \times R$ and $\mathcal{S} = \mathcal{H} \otimes \mathcal{R}$, where H and R are complete, separable metric spaces, and \mathcal{H} and \mathcal{R} are the respective Borel σ -algebras. $Q_h(\cdot|s, a)$ denotes the marginal of $Q(\cdot|s, a)$ on $h \in H$.
2. There is a fixed probability measure κ on (H, \mathcal{H}) such that for all s and a , $Q_h(\cdot|s, a)$ is absolutely continuous with respect to κ and $\alpha(\cdot|s, a)$ is the Radon-Nikodym derivative.
3. For all s , the mapping $a \rightarrow Q_h(\cdot|s, a)$ is norm continuous; that is, for all s , all a and each sequence $\{a_m\}$ of action profiles converging to a , the sequence $\{Q_h(\cdot|s, a_m)\}$ converges to $Q_h(\cdot|s, a)$ in total variation.
4. Conditional on next period's h' , the distribution of r' in next period is independent of the current state and actions. In particular, $Q_r: H \times \mathcal{R} \rightarrow [0, 1]$ is a transition probability such that for all s , all a , and all $Z \in \mathcal{S}$, we have $Q(Z|s, a) = \int_H \int_R \mathbf{1}_Z(h', r') Q_r(dr'|h') Q_h(dh'|s, a)$.
5. For κ -almost all h , $Q_r(\cdot|h)$ (abbreviated as ν_h) is absolutely continuous with respect to an atomless probability measure ν on (R, \mathcal{R}) , and $\beta(\cdot|h)$ is the Radon-Nikodym derivative.

In the following we show that the condition of a coarser transition kernel is satisfied in noisy stochastic games.

Proposition 4. *Every noisy stochastic game has a coarser transition kernel.*

Proof. Let $\lambda(Z) = \int_H \int_R \mathbf{1}_Z(h, r) \beta(r|h) d\nu(r) d\kappa(h)$ for all $Z \in \mathcal{S}$. Let $\mathcal{G} = \mathcal{H} \otimes \{\emptyset, R\}$. It is clear that $\alpha(\cdot|s, a)$ is \mathcal{G} -measurable, we need to show that \mathcal{S} has no \mathcal{G} -atom under λ .

Fix any Borel $D \subseteq S$ with $\lambda(D) > 0$. Then there is a measurable mapping ϕ from (D, \mathcal{S}^D) to (L, \mathcal{B}) such that ϕ can generate the σ -algebra \mathcal{S}^D , where L is the unit interval endowed with the Borel σ -algebra \mathcal{B} . Let $g(h, r) = h$ for each $(h, r) \in D$, $D_h = \{r : (h, r) \in D\}$ and $H_D = \{h \in H : \nu_h(D_h) > 0\}$.

Denote $g_h(\cdot) = g(h, \cdot)$ and $\phi_h(\cdot) = \phi(h, \cdot)$ for each $h \in H_D$. Define a mapping $f : H_D \times L \rightarrow [0, 1]$ as follow: $f(h, l) = \frac{\nu_h(\phi_h^{-1}([0, l]))}{\nu_h(D_h)}$. Similarly, denote $f_h(\cdot) = f(h, \cdot)$ for each $h \in H_D$. For κ -almost all $h \in H_D$, the atomlessness of ν_h implies $\nu_h \circ \phi_h^{-1}(\{l\}) = 0$ for all $l \in L$. Thus the distribution function $f_h(\cdot)$ is continuous on L for κ -almost all $h \in H_D$.

Let $\gamma(s) = f(g(s), \phi(s))$ for each $s \in D$, and $D_0 = \gamma^{-1}([0, \frac{1}{2}])$, which is a subset of D . For $h \in H_D$, let l_h be $\max\{l \in L : f_h(l) \leq \frac{1}{2}\}$ if f_h is continuous, and 0 otherwise. It is clear that when f_h is continuous, $f_h(l_h) = 1/2$. For any $E \in \mathcal{H}$, let $D_1 = (E \times R) \cap D$, and $E_1 = E \cap H_D$. If $\lambda(D_1) = 0$, then

$$\begin{aligned} \lambda(D_0 \setminus D_1) &= \lambda(D_0) = \int_{H_D} \nu_h \circ \phi_h^{-1} \circ f_h^{-1}([0, \frac{1}{2}]) d\kappa(h) \\ &= \int_{H_D} \nu_h(\phi_h^{-1}([0, l_h])) d\kappa(h) = \int_{H_D} f(h, l_h) \nu_h(D_h) d\kappa(h) \\ &= \frac{1}{2} \int_{H_D} \nu_h(D_h) d\kappa(h) = \frac{1}{2} \lambda(D) > 0. \end{aligned}$$

If $\lambda(D_1) > 0$, then

$$\begin{aligned} \lambda(D_1 \setminus D_0) &= \int_{E_1} \int_R \mathbf{1}_{D \setminus D_0}(h, r) d\nu_h(r) d\kappa(h) = \int_{E_1} \nu_h \circ \phi_h^{-1} \circ f_h^{-1}((\frac{1}{2}, 1]) d\kappa(h) \\ &= \int_{E_1} \nu_h \circ \phi_h^{-1} \circ f_h^{-1}([0, 1] \setminus [0, \frac{1}{2}]) d\kappa(h) = \frac{1}{2} \int_{E_1} \nu_h(D_h) d\kappa(h) = \frac{1}{2} \lambda(D_1) > 0. \end{aligned}$$

Hence, D is not a \mathcal{G} -atom. Therefore, \mathcal{S} has no \mathcal{G} -atom and the condition of coarser transition kernel is satisfied. \square

By Proposition 4, the existence of stationary Markov perfect equilibria in noisy stochastic games follows from Theorem 1 directly.

Theorem 3 (Duggan (2012)). *Every noisy stochastic game possesses a stationary Markov perfect equilibrium.*

Nonexistence of stationary Markov perfect equilibrium

Levy (2013, Example II) presented a concrete example of a stochastic game satisfying all the conditions as stated in Section 2 which has no stationary Markov

perfect equilibrium. The player space is $\{A, B, C, D, \theta^1, \dots, \theta^M\}$, where M is a positive integer. Players A and B have the action space $\{L, M, R\}$, each player θ^j has the action space $\{L, R\}$, players C and D have the action space $\{1, -1\}$. The state space is $S = [0, 1]$ endowed with the Borel σ -algebra \mathcal{B} .

The transitions $Q(\cdot|s, x)$ in this example is given by $Q(s, x) = (1 - \alpha(1 - s))\delta_1 + (\alpha(1 - s))Q'(s, x)$, where α is a constant in $(0, 1]$ and the probability $Q'(\cdot|s, x)$ is given by Table 1:

		Player D	
		-1	1
Player C	-1	$U(s, 1)$	$\frac{1}{2}U(s, 1) + \frac{1}{2}\delta_1$
	1	$\frac{1}{2}U(s, 1) + \frac{1}{2}\delta_1$	δ_1

Table 1

where δ_1 is the Dirac measure at 1, and $U(s, 1)$ is the uniform distribution on $[s, 1]$ for $s \in [0, 1)$.

The following proposition shows that the condition of a decomposable coarser transition kernel on the atomless part is violated in this example.¹⁵

Proposition 5. *The atomless part $(\alpha(1 - s))Q'(s, x)$ of the transition probability in Levy's Example does not have a decomposable coarser transition kernel.*

Proof. Given the state $s \in [0, 1)$ and action profile x in the previous stage, suppose that players C and D both play the strategy -1 , the transition probability in the current stage is $Q(s, x) = (1 - \alpha(1 - s))\delta_1 + \alpha(1 - s)U(s, 1)$. It is clear that $U(s, 1)$ is absolute continuous with respect to the Lebesgue measure η with the Radon-Nikodym derivative

$$q(s_1|s) = \begin{cases} \frac{1}{1-s} & s_1 \in [s, 1], \\ 0 & s_1 \in [0, s). \end{cases}$$

Suppose that the atomless part $(\alpha(1 - s))Q'(s, x)$ has a decomposable coarser transition kernel; so does $U(s, 1)$. Then, for some positive integer J , we have $q(\cdot|s) = \sum_{1 \leq j \leq J} q_j(\cdot, s)\rho_j(\cdot)$ for any $s \in [0, 1)$, where q_j is nonnegative and product measurable, ρ_j is nonnegative and integrable. Let \mathcal{G} be the minimal σ -algebra (with strong completion) with respect to which $q_j(\cdot, s)$ is measurable for all $1 \leq j \leq J$ and $s \in [0, 1)$. The condition of a decomposable coarser transition kernel implies that \mathcal{B} has no \mathcal{G} -atom; we shall show otherwise and thus derive a contradiction.

¹⁵Proposition 5 can also be implied by the nonexistence result in Levy (2013) and our Proposition 1. However, the argument in Levy (2013) is deep and difficult, while our proof explicitly demonstrates why his example fails to satisfy our sufficient condition in Proposition 1.

Denote $D_j = \{s_1 \in S : \rho_j(s_1) = 0\}$ for $1 \leq j \leq J$. Since $q(s_1|0) = 1$ for all $s_1 \in S$, we must have $\cap_{1 \leq j \leq J} D_j = \emptyset$, and hence $\eta(\cap_{1 \leq j \leq J} D_j) = 0$.

First suppose that $\eta(D_j) = 0$ for all j . Let $\bar{D} = \cup_{1 \leq j \leq J} D_j$; then $\eta(\bar{D}) = 0$. Fix $s' \in [0, 1)$. Let $E_j = \{s_1 \in S : q_j(s_1, s') = 0\}$ and $E_0 = \cap_{1 \leq j \leq J} E_j$. Then $E_j \in \mathcal{G}$ for $1 \leq j \leq J$, and hence $E_0 \in \mathcal{G}$. For any $s_1 \in [s', 1]$, since $q(s_1|s') > 0$, there exists $1 \leq j \leq J$ such that $q_j(s_1|s') > 0$, which means that $s_1 \notin E_j$ and $s_1 \notin E_0$. Hence, $E_0 \subseteq [0, s')$. For any $s_1 \in ([0, s'] \setminus \bar{D})$, we have $q(s_1|s') = 0$, and $\rho_j(s_1) > 0$ for each $1 \leq j \leq J$, which implies that $q_j(s_1|s') = 0$ for each $1 \leq j \leq J$, and $s_1 \in E_0$. That is, $([0, s'] \setminus \bar{D}) \subseteq E_0$. Hence, $\eta(E_0 \Delta [0, s']) = 0$. Therefore, $[0, s'] \in \mathcal{G}$ for all $s' \in [0, 1)$, which implies that \mathcal{G} coincides with \mathcal{B} and \mathcal{B} has a \mathcal{G} -atom $[0, 1)$. This is a contradiction.

Next suppose that $\eta(D_j) = 0$ does not hold for all j . Then there exists a set, say D_1 , such that $\eta(D_1) > 0$. Let $Z = \{K \subseteq \{1, \dots, J\} : 1 \in K, \eta(D^K) > 0\}$, where $D^K = \cap_{j \in K} D_j$. Hence, $\{1\} \in Z$, Z is finite and nonempty. Let K_0 be the element in Z containing most integers; that is, $|K_0| \geq |K|$ for any $K \in Z$, where $|K|$ is the cardinality of K . Let $K_0^c = \{1, \dots, J\} \setminus K_0$. Then K_0^c is not empty since $\eta(\cap_{1 \leq j \leq J} D_j) = 0$. In addition, $\eta(D^{K_0} \cap D_j) = 0$ for any $j \in K_0^c$. Otherwise, $\eta(D^{K_0} \cap D_j) > 0$ for some $j \in K_0^c$ and hence $(K_0 \cup \{j\}) \in Z$, which contradicts the choice of K_0 . Let $\hat{D} = \cup_{k \in K_0^c} (D^{K_0} \cap D_k)$; then $\eta(\hat{D}) = 0$. For all $s_1 \in D^{K_0}$, $q(s_1|s) = \sum_{k \in K_0^c} q_k(s_1, s) \rho_k(s_1)$ for all $s \in [0, 1)$.

Fix $s' \in [0, 1)$. Let $E_k = \{s_1 \in S : q_k(s_1, s') = 0\}$ and $E_{K_0^c} = \cap_{k \in K_0^c} E_k$. Then $E_k \in \mathcal{G}$ for any k and hence $E_{K_0^c} \in \mathcal{G}$. For any $s_1 \in [s', 1]$, since $q(s_1|s') > 0$, there exists $k \in K_0^c$ such that $q_k(s_1|s') > 0$, which means that $s_1 \notin E_k$ and $s_1 \notin E_{K_0^c}$. Hence, $E_{K_0^c} \subseteq [0, s')$, and $E_{K_0^c} \cap D^{K_0} \subseteq [0, s'] \cap D^{K_0}$. Now, for any $s_1 \in (([0, s'] \cap D^{K_0}) \setminus \hat{D})$, we have $q(s_1|s') = 0$, and $\rho_k(s_1) > 0$ for each $k \in K_0^c$, which implies that $q_k(s_1|s') = 0$ for each $k \in K_0^c$, and $s_1 \in E_{K_0^c}$. That is, $(([0, s'] \cap D^{K_0}) \setminus \hat{D}) \subseteq E_{K_0^c} \cap D^{K_0}$. Hence, $([0, s'] \cap D^{K_0}) \setminus (E_{K_0^c} \cap D^{K_0}) \subseteq \hat{D}$, and $\eta((E_{K_0^c} \cap D^{K_0}) \Delta ([0, s'] \cap D^{K_0})) = 0$. Thus, \mathcal{B} has a \mathcal{G} -atom D^{K_0} . This is again a contradiction. \square

5 Concluding Remarks

We consider stationary Markov perfect equilibria in discounted stochastic games with a general state space. So far, only several special classes of stochastic games have been shown to possess equilibria, while the existence of such equilibria under some general condition has been an *open problem*. In the literature, the standard approach for the existence arguments is to work with the convex hull of the collection of all selections from the equilibrium payoff correspondence. We

adopt this approach and provide a very simple proof of some existence results under the general condition of a (decomposable) coarser transition kernel. The minimality of our condition is illustrated. As shown in Section 4, our results strictly generalize various previous existence results and provide some explanation why a recent counterexample fails to have an equilibrium in stationary strategies as well.

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